Journal of Computer Science and Cybernetics, V.35, N.4 (2019), 293–304
DOI 10.15625/1813-9663/35/4/13898

PYTHAGOREAN PICTURE FUZZY SETS, PART 1- BASIC NOTIONS

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Abstract. Picture fuzzy set (2013) is a generalization of the Zadeh' fuzzy set (1965) and the Antanassov' intuitionistic fuzzy set. The new concept could be useful for many computational intelligent problems. Basic operators of the picture fuzzy logic were studied by Cuong, Ngan [10, 11]. New concept –Pythagorean picture fuzzy set (PPFS) is a combination of Picture fuzzy set with the Yager's Pythagorean fuzzy set [12, 13, 14]. First, in the Part 1 of this paper, we consider basic notions on PPFS as set operators of PPFS's, Pythagorean picture relation, Pythagorean picture fuzzy soft set. Next, the Part 2 of the paper is devoted to main operators in fuzzy logic on PPFS: picture negation operator, picture t-norm, picture t-conorm, picture implication operators on PPFS. As a result we will have a new branch of the picture fuzzy set theory.

Keywords. Picture Fuzzy Set; Pythagorean Picture Fuzzy Set.

1. INTRODUCTION

Recently, Bui Cong Cuong and Kreinovich (2013) first defined "picture fuzzy sets" (PFS) [8], which are a generalization of the Zadeh' fuzzy sets [1] and the Antanassov's intuitionistic fuzzy sets [3]. This concept is particularly effective in approaching the practical problems in relation to the synthesis of ideas, such as decisions making problems, voting analysis, fuzzy clustering, financial forecasting. The basic notions in the picture fuzzy sets theory were given in [9, 10]. The new basic connectives in picture fuzzy logic on PFS firstly were presented in [11, 25]. These new concepts are supporting to new computing procedures in computational intelligence problems and in other applications (see [17, 18, 19, 20, 21, 22, 23, 24]).

In 2013, Yager introduced new concept - Pythagorean fuzzy set (PFS) with some new applications in decision making problems [12, 13, 14]. This paper is devoted to Pythagorean Picture Fuzzy set (PPFS) - a combination of Picture fuzzy set with the Pythagorean fuzzy set. First, in first section, we present basic notions on PPFS as set operators and Cartesian product of PPFS's, Pythagorean picture relation, Pythagorean picture fuzzy soft set.

2. BASIC NOTIONS OF PYTHAGOREAN PICTURE FUZZY SET

We first define basic notions of Pythagorean picture fuzzy sets.

Definition 2.1. [8] A picture fuzzy set A on a universe U is an object of the form

$$A = \{(u, x_{1A}(u), x_{2A}(u), x_{3A}(u))\,|u \in U\},$$

where $x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of u in A, and the following conditions are satisfied

$$0 \le x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u) \le 1 \text{ and } 0 \le x_{1A}(u) + x_{2A}(u) + x_{3A}(u) \le 1, \forall u \in U.$$

Then, $\forall u \in U$, $x_{4A}(u) = 1 - (x_{1A}(u) + x_{2A}(u) + x_{3A}(u))$ is called the degree of refusal membership of u in A.

Definition 2.2. A Pythagorean picture fuzzy set (PPFS) A on a universe U is an object of the form $A = \{(u, x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u))\,|u \in U\}$, where $x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of u in A, and the following conditions are satisfied

$$0 \le x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u) \le 1 \ \text{ and } \ 0 \le x_{1A}^2(u) + x_{2A}^2(u) + x_{3A}^2(u) \le 1, \forall u \in U.$$

Consider the sets

$$D^* = \{x = (x_1, x_2, x_3)|x \in [0,1]^3,\ x_1 + x_2 + x_3 \le 1\},$$
$$P^* = \{x = (x_1, x_2, x_3)|x \in [0,1]^3, x_1^2 + x_2^2 + x_3^2 \le 1\}.$$
$$0_{D^*} = 0_{P^*} = (0,0,1) \in P^*,\ 1_{D^*} = 1_{P^*} = (1,0,0) \in P^*,\ \text{and } D^* \subseteq P^*.$$

From now on, we will assume that if $x \in P^*$, then x_1, x_2 and x_3 denote, respectively, the first, the second and the third component of x, i.e., $x = (x_1, x_2, x_3)$.

We have a lattice $(P^*, \le_1)$, where $\le_1$ defined by $\forall x, y \in P^*$

$$(x \le_1 y) \Leftrightarrow (x_1 < y_1, x_3 \ge y_3) \vee (x_1 = y_1, x_3 > y_3) \vee (\{x_1 = y_1, x_3 = y_3, x_2 \le y_2\}),$$

$$(x = y) \Leftrightarrow (x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3),\ \forall x,\ y \in P^*.$$

We define the first, second and third projection mapping pr_1, then pr_2 and pr_3 on P^*, defined as $pr_1(x) = x_1$ and $pr_2(x) = x_2$ and $pr_3(x) = x_3$, on all $x \in P^*$.

Note that, if for $x, y \in P^*$ that neither $x \le_1 y$ nor $y \le_1 x$, then x and y are incomparable w.r.t $\le_1$, denoted as $x\|_{\le_1} y$.

From now on, we denote $u \wedge v = \min(u, v),\ u \vee v = \max(u, v)$ for all $u, v \in R^1$.

For each $x, y \in P^*$, we define

$$\inf(x,y) = \begin{cases} \min(x,y), \text{ if } x \le_1 y \text{ or } y \le_1 x \\ (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3,\ x_3 \vee y_3), \text{ else,} \end{cases}$$

$$\sup(x,y) = \begin{cases} \max(x,y), \text{ if } x \le_1 y \text{ or } y \le_1 x \\ (x_1 \vee y_1, 0, x_3 \wedge y_3), \text{ else.} \end{cases}$$

Proposition 2.1. *With these definitions $(P^*, \le_1)$ is a complete lattice.*

Proof. See [11].

Using this lattice, we easily see that every Pythagorean picture fuzzy set

$$A = \{(u,\ x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u))\,|u \in U\},$$

corresponds an P^*- fuzzy set [11] mapping, i.e., we have a mapping

$$A : U \to P^* : \; u \to (x_{1A}(u), \; x_{2A}(u), \; x_{3A}(u)) \in P^*.$$

Interpreting Pythagorean picture fuzzy sets as P^*- fuzzy sets gives way to greater flexibility in calculating with membership degrees, since the triplet of numbers formed by the three degrees is an element of P^*, and often allows to obtain more compact formulas.

Let $PFS(U)$ denote the set of all the picture fuzzy set PFSs on a universe U and $PPFS(U)$ denote the set of all Pythagorean picture fuzzy set PPFSs on a universe U.

Definition 2.3. For every two PPFSs A and B, $B = \{(u, \; x_{1B}(u), \; x_{2B}(u), \; x_{3B}(u)) \, | u \in U\}$, the inclusion, union, intersection and complement are defined as follows

$$A \subseteq B \text{ iff } (\forall u \in U, \; x_{1A}(u) \le x_{1B}(u) \text{ and } x_{2A}(u) \le x_{2B}(u) \text{ and } x_{3A}(u) \ge x_{3B}(u)),$$

$$A = B \text{ iff } (A \subseteq B \text{ and } B \subseteq A),$$

$$A \cup B = \{(u, \; x_{1A}(u) \vee x_{1B}(u), \; x_{2A}(u) \wedge x_{2B}(u), \; x_{3A}(u) \wedge x_{3B}(u)) \, | u \in U\},$$

$$A \cap B = \{(u, \; x_{1A}(u) \wedge x_{1B}(u)), \; x_{2A}(u) \wedge x_{2B}(u), \; x_{3A}(u) \vee x_{3B}(u)) \, | u \in U\}$$

$$coA = A^c = \left\{(u, \; x_{3A}(u), \sqrt{1 - (x_{1A}^2(u) + x_{2A}^2(u) +} x_{3A}^2(u)), \; x_{1A}(u)) \, | u \in U\right\}.$$

Now we consider some propeties of the defined operations on PPFS.

Proposition 2.2. *For every PPFS's A,B,C*

(a) *If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$;*

(b) $(A^c)^c = A$;

(c) *Operations $\cap$ and $\cup$ are commutative, associative, and distributive.*

The detail proof see [26].

Convex combination is an important operation in mathematics, which is a useful tool on convex analysis, linear spaces and convex optimization.

Definition 2.4. Let A, B be two PPFS on U. Let θ be a real number such that $0 \le \theta \le 1$.

For each θ, the convex combination of A and B is defined as follows

$$C_\theta(A, B) = \{(u, \; x_{1C_\theta}(u), \; x_{2C_\theta}(u), \; x_{3C_\theta}(u)) \, | u \in U\}$$

where $\forall u \in U$,

$$x_{1C_\theta}(u) = \theta.x_{1A}(u) + (1 - \theta).x_{1B}(u),$$

$$x_{2C_\theta}(u) = \theta.x_{2A}(u) + (1 - \theta).x_{2B}(u),$$

$$x_{3C_\theta}(u) = \theta.x_{3A}(u) + (1 - \theta).x_{3B}(u).$$

Proposition 2.3. *Let A, B be two PPFS on U. Let θ be a real number such that $0 \le \theta \le 1$. Then*

If $\theta = 1$, then $C_\theta(A, B) = A$ and if $\theta = 0$, then $C_\theta(A, B) = B$;

If $A \subseteq B$, then $\forall \theta$, $A \subseteq C_\theta(A, B) \subseteq B$;

If $(A \supseteq B) \& (\theta_1 \ge \theta_2)$, then $C_{\theta_1}(A, B) \supseteq C_{\theta_2}(A, B)$.

Definition 2.5. Let U_1 and U_2 be two universums and let

$$A = \{(u,\ x_{1A}(u),\ x_{2A}(u),\ x_{3A}(u))\,|u \in U_1\}$$

and

$$B = \{(v,\ x_{1B}(v),\ x_{2B}(v),\ x_{3B}(v))\,|v \in U_2\},$$

be two PPFSs. We define the Cartesian product of these two PPFS's

$$A \times B = \{((u,v),\ x_{1A}(u) \wedge x_{1B}(v),\ x_{2A}(u) \wedge x_{2B}(v),\ x_{3A}(u) \wedge x_{3B}(v))\,|(u,v) \in U_1 \times U_2\}.$$

We denote the set of all PPFS over $X_1 \times X_2$ by $PPFS(X_1 \times X_2)$.

Theorem 2.1. *For every three universums U_1, U_2, U_3 and four PPFSs $O_1, O_2 \in PPFS(U_1)$, $O_3 \in PPFS(U_2)$, $O_4 \in PPFS(U_3)$. We have the following properties of Cartesian productions on PPFS*

(a) $O_1 \times O_3 = O_3 \times O_1$;
(b) $(O_1 \times O_3) \times O_4 = O_1 \times (O_3 \times O_4)$;
(c) $(O_1 \cup O_2) \times O_3 = (O_1 \times O_3) \cup (O_2 \times O_3)$;
(d) $(O_1 \cap O_2) \times O_3 = (O_1 \times O_3) \cap (O_2 \times O_3)$.

Proof. We omitt the proof (a), (b).

(c) $O_1, O_2 \in PPFS(X_1)$, then
$O_1 = \{(u, x_{1O_1}(u), x_{2O_1}(u), x_{3O_1}(u))\,|u \in X_1\}$,
$O_2 = \{(u, x_{1O_2}(u), x_{2O_2}(u), x_{3O_2}(u))\,|u \in X_1\}$,
and

$$O_1 \cup O_2 = \{(u,\ x_{1O_1}(u) \vee x_{1O_2}(u),\ x_{2O_1}(u) \wedge x_{2O_2}(u),\ x_{3O_1}(u) \wedge x_{3O_2}(u))\,|u \in X_1\},$$

$$(O_1 \cup O_2) \times O_3 = \left\{ \begin{array}{l} (u,v), (x_{1(O_1 \cup O_2)}(u) \wedge x_{1O_3}(v), \\ x_{2(O_1 \cup O_2)}(u) \wedge x_{2O_2}(v), x_{3(O_1 \cup O_2)}(u) \wedge x_{3O_3}(v))\,|(u,v) \in X_1 \times X_2 \end{array} \right\}.$$

Using the properties of the operations $\wedge$ and $\vee$ and for all $u \in X_1,\ v \in X_2$ we have

$$\begin{array}{l} (O_1 \cup O_2) \times O_3 = \\ \{((u,v), x_{1(O_1 \cup O_2)}(u) \wedge x_{1O_3}(v), x_{2(O_1 \cup O_2)}(u) \wedge x_{2O_3}(v), x_{3(O_1 \cup O_2)}(u) \wedge x_{3O_3}(v)))\} = \\ \{((u,v), (x_{1O_1}(u) \vee x_{1O_2}(u)) \wedge x_{1O_3}(v), (x_{2O_1}(u) \wedge x_{2O_2}(u) \wedge x_{2O_3}(v)), \\ (x_{3O_1}(u) \wedge x_{3O_2}(u)) \wedge x_{3O_3}(v)))\}. \end{array}$$

$$\begin{aligned} x_{1(O_1 \cup O_2) \times O_3}(u,v) &= (x_{1O_1}(u) \vee x_{1O_2}(u)) \wedge x_{1O_3}(v)) \\ &= (x_{1O_1}(u) \wedge x_{1O_3}(v)) \vee (x_{1O_2}(u) \wedge x_{1O_3}(v)) \\ &= x_{1(O_1 \times O_3) \cup (O_2 \times O_3)}(u,v),\ \forall u \in X_1, v \in X_2 \end{aligned}$$

$$\begin{aligned} x_{2(O_1 \cup O_2) \times O_3}(u,v) &= (x_{2O_1}(u) \wedge x_{2O_2}(u)) \wedge x_{2O_3}(v)) \\ &= (x_{2O_1}(u) \wedge x_{2O_3}(v)) \wedge (x_{2O_2}(u) \wedge x_{2O_3}(v)) \\ &= x_{2(O_1 \times O_3) \cup (O_2 \times O_3)}(u,v),\ \forall u \in X_1, v \in X_2 \end{aligned}$$

$$\begin{aligned} x_{3(O_1\cup O_2)\times O_3}(u,v) &= (x_{3O_1}(u)\wedge x_{3O_2}(u))\wedge x_{3O_3}(v)) \\ &= (x_{3O_1}(u)\wedge x_{3O_3}(v))\wedge (x_{3O_2}(u)\wedge x_{3O_3}(v)) \\ &= x_{3(O_1\times O_3)\cup(O_2\times O_3)}(u,v),\ \forall u\in X_1, v\in X_2 \end{aligned}$$

The proof is given.

(d) The proof is analogous. ∎

Fuzzy relations were defined and used in Fuzzy control. The Zadeh' composition rule of inference (see [2, 5, 7]) is a well-known method in approximation theory and fuzzy relations were used in these inference methods in fuzzy systems.

Let X, Y and Z be ordinary non-empty sets.

An extension the results given in [5, 6, 7] for PPFS is the following.

Definition 2.6. A Pythagorean picture fuzzy relation is a Pythagorean picture fuzzy subset of $X \times Y$, i.e. R given by

$$R = \{((x,y), z_{1R}(x,y),\ z_{2R}(x,y),\ z_{3R}(x,y))\,|x\in X,\ y\in Y)\},$$

where $z_{1R} : X \times Y \to [0,1]$, $z_{2R} : X \times Y \to [0,1]$, $z_{3R} : X \times Y \to [0,1]$ satisfy the condition $0 \le z_{1R}^2(x,y) + z_{2R}^2(x,y) + z_{3R}^2(x,y) \le 1$ for every $(x,y) \in (X \times Y)$.

We will denote by $PPFR(X \times Y)$ the set of all the Pythagorean picture fuzzy subsets in $X \times Y$.

A generalization of the composition of fuzzy relations [5] is the following.

The first composition of PPFRs is the generalized min-max composition in fuzzy set theory.

Definition 2.7. [9] Let $E \in PPFR(X\times Y)$ and $P \in PPFR(Y \times Z)$. We will call max-min composition of relation E and relation P is defined as follow, where $\forall(x,z) \in (X \times Z)$,

$$PCE = \{((x,z), x_{1PCE}(x,z), x_{2PCE}(x,z), x_{3PCE}(x,z))\,|x\in X, z\in Z)\},\ \forall(x,z)\in X\times Z,$$

$$x_{1PCE}(x,z) = \bigvee_y \{[x_{1E}(x,y)\wedge x_{1P}(y,z)]\},$$

$$x_{2PCE}(x,z) = \bigvee_y \{[x_{2E}(x,y)\wedge x_{2P}(y,z)]\},$$

$$x_{3PCE}(x,z) = \bigwedge_y \{[x_{3E}(x,y)\vee x_{3P}(y,z)]\}.$$

3. PYTHAGOREAN PICTURE FUZZY SOFT SET

Molodtsov [15] defined the soft set in the following way. Let U be an initial universe of objects and E be the set of parameters in relation to objects in U. Parameters are often attributes, characteristics, or properties of objects. Let $P(U)$ denotes the power set of U and $A \subseteq E$.

Definition 3.1. ([15]) A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

In other words, the soft set is not a kind of set, but a parameterized family of sufsets of U. For any parameter $e \in E$, $F(e) \subseteq U$ may be considerd as the set of $e-$approximate elements of the soft set (F, A).

Maji et al. [16] initiated the study on hybrid structures involving both fuzzy set and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of (crisp) soft set. Maji et al [16] proposed the concept of the fuzzy soft sets as follows.

Definition 3.2. ([16]) Let $F(U)$ be the set of all fuzzy subsets of U. Let E the set of parameters and $A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F : A \to F(U)$.

Definition 3.3. Let $PPFS(U)$ be the set of all Pythagorean picture fuzzy subsets of U. Let E be the set of parameters and $A \subseteq E$. A pair (F, A) is called a Pythagorean picture fuzzy soft set over U, where F is a mapping given by $F : A \to PPFS(U)$.

Clearly, for any parameter $e \in A$, $F(e)$ can be written as a Pythagorean picture fuzzy set such that $F(e) = \{(u,\ x_{1F(e)}(u),\ x_{2F(e)}(u),\ x_{3F(e)}(u))\, |u \in U\}$.

We denote the set of all Pythagorean picture fuzzzy soft sets over U by $PPfss(U)$.

Example 3.1. Consider a Pythagorean picture fuzzy soft set (F, A), where U is the set of four economic projects under the consideration of a decision committee to choose, which is denoted by $U = \{p_1, p_2, p_3, p_4\}$, and A is a parameter set, where $A = \{e_1, e_2, e_3, e_4, e_5\}=$ {good finance indicator, average finance indicator, good social contribution, average social contribution, good environment indicator}. The Pythagorean picture fuzzy soft set $\langle F, A\rangle$ describes the "attractiveness of the projects" to the decision committee.

Suppose that:

$$F(e_1) = \{(p_1, 0.8, 0.12, 0.05), (p_2, 0.9, 0.18, 0.16), (p_3, 0.55, 0.20, 0.21), (p_4, 0.50, 0.20, 0.24)\},$$

$$F(e_2) = \{(p_1, 0.82, 0.05, 0.10), (p_2, 0.7, 0.12, 0.10), (p_3, 0.60, 0.14, 0.10), (p_4, 0.82, 0.10, 0.24)\},$$

$$F(e_3) = \{(p_1, 0.60, 0.14, 0.16), (p_2, 0.55, 0.20, 0.16), (p_3, 0.70, 0.15, 0.11), (p_4, 0.63, 0.12, 0.18)\},$$

$$F(e_4) = \{(p_1, 0.86, 0.12, 0.07), (p_2, 0.75, 0.05, 0.16), (p_3, 0.60, 0.17, 0.18), (p_4, 0.55, 0.10, 0.22)\},$$

$$F(e_5) = \{(p_1, 0.60, 0.12, 0.07), (p_2, 0.62, 0.14, 0.16), (p_3, 0.55, 0.10, 0.21), (p_4, 0.70, 0.20, 0.05)\}.$$

The Pythagorean picture fuzzy soft set (F, A) is a parameterized family $\{F(e_i) : i = 1, 2, 3, 4, 5\}$ of Pythagorean picture fuzzy sets over U.

Now we give some properties of these new sets.

Definition 3.4. For two Pythagorean picture fuzzy soft sets (F, A) and (G, B) over a commom universe U, we say that (F, A) is a Pythagorean picture fuzzy soft subset of (G, B), denoted $(F, A) \subseteq (G, B)$, if it is satisfies $A \subseteq B$ and $F(e) \subseteq G(e)$, $\forall e \in A$.

Similary (F, A) is called a superset of (G, B) if (G, B) is a soft subset of (F, A). This relation is denoted by $(F, A) \supseteq (G, B)$.

Definition 3.5. For two Pythagorean picture fuzzy soft sets (F, A) and (G, B) over a commom universe U are called soft equal if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

We write $(F, A) = (G, B)$. In this case $A = B$ and $F(e) = G(e)$, $\forall e \in A$.

Some operations and properties of Pythagorean picture fuzzy soft sets.

Now we define some operations on Pythagorean picture fuzzy soft sets and present some properties.

Definition 3.6. The complement of a Pythagorean picture fuzzy soft set (F, A) is denoted as $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to P(U)$ is a mapping given by $F^c(e) = (F(e))^c$, for all $e \in A$.

Definition 3.7. If (F, A) and (G, B) are two Pythagorean picture fuzzy soft sets over a commom universe U, then "(F, A) and (G, B)", is a Pythagorean picture fuzzy soft set denoted by $(F, A) \wedge (G, B)$ and it is defined by $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$ for all $(\alpha, \beta) \in A \times B,\ u \in U$, that is

$$H(\alpha, \beta)(u) = (x_{1F(\alpha)}(u) \wedge x_{1G(\beta)}(u), x_{2F(\alpha)}(u) \wedge x_{2G(\beta)}(u), x_{3F(\alpha)}(u) \vee x_{3G(\beta)}(u)).$$

Definition 3.8. If (F, A) and (G, B) are two Pythagorean picture fuzzy soft sets over a commom universe U, then "(F, A) or (G, B)" is a Pythagorean picture fuzzy soft set denoted by $(F, A) \vee (G, B)$ is defined by $(F, A) \vee (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cup G(\beta)$ for all $(\alpha, \beta) \in A \times B,\ u \in U$, that is

$$H(\alpha, \beta)(u) = (x_{1F(\alpha)}(u) \vee x_{1G(\beta)}(u), x_{2F(\alpha)}(u) \wedge x_{2G(\beta)}(u), x_{3F(\alpha)}(u) \wedge x_{3G(\beta)}(u)).$$

Theorem 3.1. *Let (F, A), (G, B) and (H, C) be three Pythagorean picture fuzzy soft sets over U, then we have the following properties:*

(1) $(F, A) \wedge ((G, B) \wedge (H, C)) = ((F, A) \wedge (G, B)) \wedge (H, C)$;

(2) $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$.

Proof. (1). Assume that $(G, B) \wedge (H, C) = (I, B \times C)$, where $I(\beta, \gamma) = G(\beta) \cap H(\gamma),\ \forall (\beta, \gamma) \in B \times C$. Thus, we have

$$I(\beta, \gamma)(u) = (x_{1G(\beta)}(u) \wedge x_{1H(\gamma)}(u), x_{2G(\beta)}(u) \wedge x_{2H(\gamma)}(u), x_{3G(\beta)}(u) \vee x_{3H(\gamma)}(u)),$$

$\forall (\beta, \gamma) \in B \times C,\ u \in U$.

Since $(F, A) \wedge ((G, B) \wedge (H, C)) = (F, A) \wedge (I, B \times C)$, we suppose that
$(F, A) \wedge (I, B \times C) = (K, A \times B \times C)$,
$K(\alpha, \beta, \gamma) = F(\alpha) \cap I(\beta, \gamma)$,
$(\alpha, \beta, \gamma) \in A \times (B \times C) = A \times B \times C$.

Hence

$$\begin{aligned} K(\alpha, \beta, \gamma)(u) &= (F(\alpha) \cap I(\beta, \gamma))(u) \\ &= \left(x_{1F(\alpha)}(u) \wedge x_{1I(\beta,\gamma)}(u), x_{2F(\alpha)}(u) \wedge x_{2I(\beta,\gamma)}(u), x_{3F(\alpha)}(u) \vee x_{3I(\beta,\gamma)}(u)\right) \\ &= \big(x_{1F(\alpha)}(u) \wedge_{1G(\beta)} (u) \wedge x_{1H(\gamma)}(u), x_{2F(\alpha)}(u) \wedge x_{2G(\beta)}(u) \wedge x_{2H(\gamma)}(u), \\ &\qquad x_{2F(\alpha)}(u) \vee x_{2G(\beta)}(u) \vee x_{3H(\gamma)}(u)\big). \end{aligned}$$

Now we assume that $(F, A) \wedge (G, B) = (J, A \times B)$, where $J(\alpha, \beta) = F(\alpha) \cap G(\beta),\ \forall (\alpha, \beta) \in A \times B$.

Thus, we have

$$J(\alpha, \beta)(u) = (x_{1F(\alpha)}(u) \wedge x_{1G(\beta)}(u)), x_{2F(\alpha)}(u) \wedge x_{2G(\beta)}(u)), x_{3F(\alpha)}(u) \vee x_{3G(\beta)}(u)),$$

$\forall (\alpha, \beta) \in A \times B,\ u \in U$.

Since $((F, A) \wedge (G, B)) \wedge (H, C)) = (J, A \times B) \wedge (H, C)$, we suppose that

$(J, A \times B) \wedge (H, C) = (K^1, A \times B \times C),$
$K^1(\alpha, \beta, \gamma) = J(\alpha, \beta) \wedge H(\gamma),$
$(\alpha, \beta, \gamma) \in A \times (B \times C) = A \times B \times C.$

Hence

$$\begin{aligned} K^1(\alpha,\beta,\gamma)(u) &= \left(J(\alpha,\beta) \cap H(\gamma)\right)(u) \\ &= \big(x_{1J(\alpha,\beta)}(u) \wedge x_{1H(\gamma)}(u), x_{2J(\alpha,\beta)}(u) \wedge x_{2H(\gamma)}(u), x_{3J(\alpha,\beta)}(u) \vee x_{3H(\gamma)}(u)\big) \\ &= \big(x_{1F(\alpha)}(u) \wedge x_{1G(\beta)}(u) \wedge x_{1H(\gamma)}(u), x_{2F(\alpha)}(u) \wedge x_{2G(\beta)}(u) \wedge x_{2H(\gamma)}(u)), \\ &\quad x_{3F(\alpha)}(u) \vee x_{3G(\beta)}(u) \vee x_{3H(\gamma)}(u)\big) \\ &= K(\alpha,\beta,\gamma)(u) \ (\alpha,\beta,\gamma) \in A \times B \times C, \ u \in U. \end{aligned}$$

Consequently, K and K^1 are the same operations. Thus $(F, A) \wedge ((G, B) \wedge (H, C)) = ((F, A) \wedge (G, B)) \wedge (H, C)$.

The proof of (2) is analogous. ■

Definition 3.9. The *intersection* of two Pythagorean picture fuzzy soft sets (F, A) and (G, B) over a commom universe U is denoted by $(F, A) \wedge_1 (G, B)$, which is a Pythagorean picture fuzzy soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if} \quad e \in A - B, \\ G(e) & \text{if} \quad e \in B - A, \\ F(e) \cap G(e) & \text{if} \quad e \in A \cap B. \end{cases}$$

It means, $\forall e \in A \cap B$ then

$$H(e) = \{(u, x_{1F(e)}(u) \wedge x_{1G(e)}(u), x_{2F(e)}(u) \wedge x_{2G(e)}(u), x_{3F(e)}(u) \vee x_{3G(e)}(u)) \,|u \in U\}.$$

Definition 3.10. The *union* of two Pythagorean picture fuzzy soft sets (F, A) and (G, B) over a commom universe U is denoted by $(F, A) \vee_1 (G, B)$, which is a Pythagorean picture fuzzy soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if} \quad e \in A - B, \\ G(e) & \text{if} \quad e \in B - A, \\ F(e) \cup G(e) & \text{if} \quad e \in A \cap B. \end{cases}$$

It means, $\forall e \in A \cap B$ then

$$H(e) = \{(u, x_{1F(e)}(u) \vee x_{1G(e)}(u), x_{2F(e)}(u) \wedge x_{2G(e)}(u), x_{3F(e)}(u) \wedge x_{3G(e)}(u)) \,|u \in U\}.$$

Theorem 3.2. *Let (F, A), (G, B) and (H, C) be three Pythagorean picture fuzzy soft sets over U, then we have the following properties:*

(1) $(F, A) \wedge_1 ((G, B) \wedge_1 (H, C)) = ((F, A) \wedge_1 (G, B)) \wedge_1 (H, C)$;
(2) $(F, A) \vee_1 ((G, B) \vee_1 (H, C)) = ((F, A) \vee_1 (G, B)) \vee_1 (H, C)$.

Now we give the definion of the Cartesian product of Pythagorean picture fuzzy soft sets.

Definition 3.11. Let U_1 and U_2 be two universums and let E be the set of parameters and $A, B \subseteq E$. Let (F, A), (G, B) be two Pythagorean picture fuzzy soft set over U_1, U_2, corresponding. Then the Cartesian product $(F, A) \times (G, B)$ is a Pythagorean picture fuzzy soft set over $U_1 \times U_2$ is defined by $(F, A) \times (G, B) = (H, A \times B)$, where

$$H(\alpha, \beta)(u, v) = (x_{1F(\alpha)}(u) \wedge x_{1G(\beta)}(v), x_{2F(\alpha)}(u) \wedge x_{2G(\beta)}(v)), x_{3F(\alpha)}(u) \wedge x_{3G(\beta)}(v))$$

$\forall(\alpha, \beta) \in A \times B,\ \forall u \in U_1, v \in U_2$.

Theorem 3.3. *Let U_1, U_2, U_3 be three universums and let E be the set of parameters and $A_1, A_2, B, D \subseteq E$ and four Pythagorean picture fuzzy soft sets $(F_1, A_1), (F_2, A_2) \in PPfss(U_1)$, $(G, B) \in PPfss(U_2)$, $(H, D) \in PPfss(U_3)$:*

(a) $\langle F_1, A_1\rangle \times \langle G, B\rangle = \langle G, B\rangle \times \langle F_1, A_1\rangle$;

(b) $(\langle F_1, A_1\rangle \times \langle G, B\rangle) \times \langle H, D\rangle = \langle F_1, A_1\rangle \times (\langle G, B\rangle \times \langle H, D\rangle)$;

(c) $(\langle F_1, A_1\rangle \cup \langle F_2, A_2\rangle) \times \langle G, B\rangle = (\langle F_1, A_1\rangle \times \langle G, B\rangle) \cup (\langle F_2, A_2\rangle \times \langle G, B\rangle)$;

(d) $(\langle F_1, A_1\rangle \cap \langle F_2, A_2\rangle) \times \langle G, B\rangle = (\langle F_1, A_1\rangle \times \langle G, B\rangle) \cap (\langle F_2, A_2\rangle \times \langle G, B\rangle)$.

4. FINAL CONCLUSION AND FUTURE WORK

In this paper we give the definition of Pythagorean Picture fuzzy set – a combination of the concept of Picture Fuzzy set with the concept of Yager 's Pythagorean fuzzy set and consider basic notions of the new sets. Some properties of some new definitions were presented to construct a new branch of Picture Fuzzy Set Theory, which should be useful to practical computational intelligent problems. As Yager in [13, 14] remarked that the new model could useful for new practical problems. In the future papers we should present main connectives in fuzzy logic on PPFS, which provided tools for new problems in picture fuzzy systems.

ACKNOWLEDGMENT

This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.01-2017.02.

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Received on June 26, 2019
Revised on August 29, 2019